

Spinori na zakrivljenom nekomutativnom prostoru

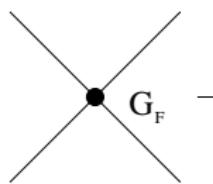
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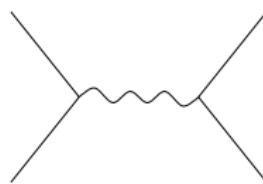
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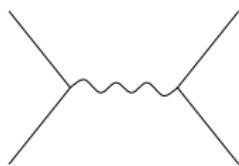
Problemi gravitacije kao kvantne teorije polja



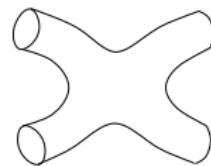
Fermijeva teorija



Elektroslaba teorija



Razmena gravitona

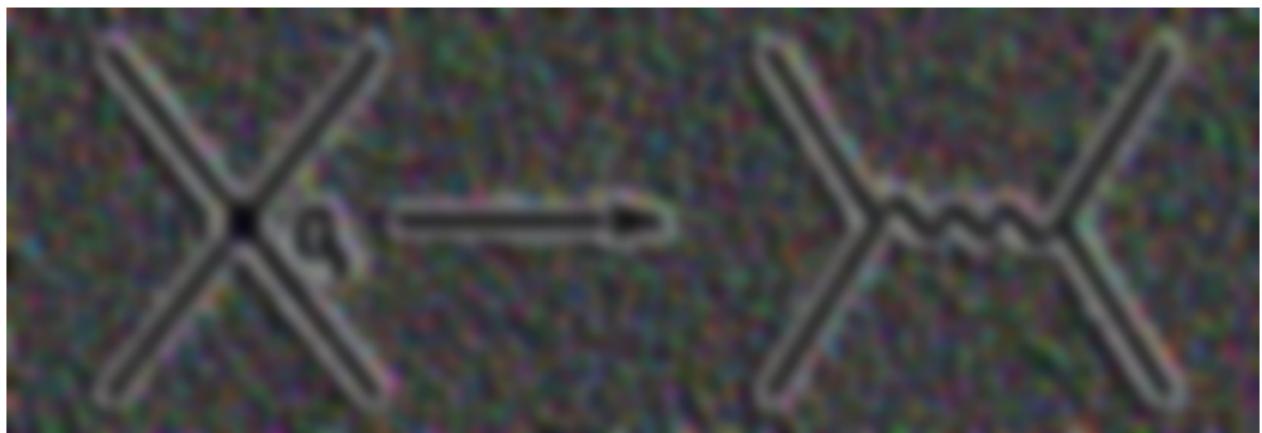


Struna

Nekomutativna geometrija

$$[x^\mu, x^\nu] = i\hbar J^{\mu\nu}. \quad (1)$$

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad \theta^{\mu\nu} = \text{const.} \quad (2)$$



Nepostojanje tačke na Plankovij skali

Spinorska reprezentacija u N dimenzija

- N matrica γ_μ koje zadovoljavaju:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\mathbb{1}, \quad \mu, \nu = 1, \dots, N. \quad (3)$$

- Matrica $\mathbb{1}$ je iste dimenzije.
- Uslov ermitnosti:

$$\gamma_\mu^\dagger = \gamma_\mu. \quad (4)$$

- $SO(N)$ generatori:

$$\Sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu] = -\Sigma_{\nu\mu}. \quad (5)$$

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = \delta_{\mu\rho}\Sigma_{\nu\sigma} + \delta_{\nu\rho}\Sigma_{\mu\sigma} - \delta_{\mu\sigma}\Sigma_{\nu\rho} - \delta_{\nu\sigma}\Sigma_{\mu\rho}. \quad (6)$$

- $SO(N)$ spinori su objekti koji se pri $SO(N)$ transformišu kao:

$$\delta\psi = \frac{1}{2}\theta^{\mu\nu}\Sigma_{\mu\nu}\psi \quad (7)$$

- Spinori su kolone iste dimenzije kao γ -matrice.
- Zadatak: Naći reprezentaciju γ -matrica

Reprezentacija γ -matrica u parnom broju dimenzija

- Za $N = 2n$, uvedemo matrice:

$$b_i = \frac{1}{\sqrt{2}}(\gamma_i + i\gamma_{i+n}), \quad b_i^\dagger = \frac{1}{\sqrt{2}}(\gamma_i - i\gamma_{i+n}). \quad i = 1, \dots, n \quad (8)$$

- Algebra fermionskog oscilatora:

$$\{b_i^\dagger, b_j\} = \delta_{ij}, \quad \{b_i^\dagger, b_j^\dagger\} = \{b_i, b_j\} = 0 \quad (9)$$

- Dvodimenzionalni Hilbertov prostor za svako i -operatori krecije i anihilacije:

$$b|0\rangle = 0, \quad b^\dagger|0\rangle = |1\rangle, \quad b|1\rangle = |0\rangle, \quad b^\dagger|1\rangle = 0, \quad (10)$$

- $|0\rangle$ je vakuum, a $|1\rangle$ prvo pobudjeno stanje.
- Ukupan Hilbertov prostor razapet vektorima:

$$|0\rangle, b_i^\dagger|0\rangle, b_{i_1}^\dagger b_{i_2}^\dagger|0\rangle, \dots, b_1^\dagger \cdots b_n^\dagger|0\rangle.$$

- Jedno vakumsko stanje $|0\rangle$ annihilirano svim b_i , n stanja $b_i^\dagger |0\rangle$ dobijenih dejstvom i operatora kreacije na vakuum.
- Zbog antikomutiranja b i b^\dagger operatora, postoji $\frac{1}{2}n(n-1)$ stanja koja se dobijaju dejstvom ovih operatora
- Zbog **Grasmanovske prirode ovih operatora (Paulijev Princip)**, $(b^\dagger)^2 = 0$, postoji konačan broj stanja. Najviše stanje je: $b_1^\dagger \cdots b_n^\dagger |0\rangle$. Ukupan broj stanja:

$$1 + n + \frac{1}{2}n(n-1) + \cdots + 1 = \sum_{p=1}^n \binom{n}{p} = 2^n \quad (11)$$

- Invertovanjem definicije za b operatore:

$$\gamma_i = \frac{1}{\sqrt{2}}(b_i^\dagger + b_i), \quad \gamma_{i+n} = \frac{i}{\sqrt{2}}(b_i^\dagger - b_i) \quad (12)$$

konstruisali smo jednu reprezentaciju $2^n \times 2^n$ -dimenzionalnih γ -matrica (do na $\gamma'_\mu = S^{-1}\gamma_\mu S$) koje deluju na **spinore**

$$\psi = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^{2^n} \end{pmatrix}.$$

Matrica kiralnosti

- Postoji još jedna linearno nezavisna matrica:

$$\gamma = \alpha \gamma_1 \dots \gamma_N, \quad \gamma^\dagger = \gamma, \quad \gamma^2 = \mathbb{1}. \quad (13)$$

$$\begin{aligned}\gamma^2 &= \alpha^2 \gamma_1 \dots \gamma_N \gamma_1 \dots \gamma_N = \alpha^2 (-1)^{\frac{(N-1)N}{2}} = \mathbb{1} \\ \alpha &= (-1)^{\frac{N-1}{2}}, N = 2n\end{aligned}$$

- Ova matrica komutira sa svim ostalim γ -matricama:

$$\{\gamma_\mu, \gamma\} = \gamma_\mu \gamma + \alpha \gamma_1 \dots \gamma_N \gamma_\mu = \gamma_\mu \gamma + (-1)^{N-1} \gamma_\mu \alpha \gamma_1 \dots \gamma_N$$

$$\{\gamma, \gamma_\mu\} = 0. \quad (14)$$

- $\Rightarrow \gamma$ komutira sa $SO(N)$ generatorima:

$$[\gamma, \Sigma_{\mu\nu}] = 0, \quad \Sigma_{\mu\nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu]. \quad (15)$$

- $\gamma^2 = \mathbb{1} \Rightarrow$ svojstvene vrednosti γ su ± 1 . Postoje dva svojstvena vektora:

$$\gamma \psi_+ = \psi_+, \quad \gamma \psi_- = -\psi_- \quad (16)$$

Invarijantni potprostori

- Pri $SO(N)$ transformacijama $\delta\psi = \frac{1}{2}\theta^{\mu\nu}\Sigma_{\mu\nu}\psi$:

$$\delta\psi_{\pm} = \delta\gamma\psi = \frac{1}{2}\theta^{\mu\nu}\gamma\Sigma_{\mu\nu}\psi = \frac{1}{2}\theta^{\mu\nu}\Sigma_{\mu\nu}\gamma\psi = \frac{1}{2}\theta^{\mu\nu}\Sigma_{\mu\nu}\psi_{\pm} \quad (17)$$

- \implies nema mešanja svojstvenih vektora
- Ovo važi i za $SO(1, N - 1)$ Lorentzove transformacije u prostoru Minkovskog

$$\psi_+ \longrightarrow \psi_+, \quad \psi_- \longrightarrow \psi_-.$$

- \implies Reprezentacija je reducibilna
- Cikličnost traga: $\text{tr}\gamma = \text{tr}(\alpha\gamma_1 \cdots \gamma_N) = \text{tr}(\alpha\gamma_N\gamma_1 \cdots \gamma_{N-1}) = -\text{tr}\gamma$:

$$\text{tr}\gamma = 0. \quad (18)$$

- \implies Jednak broj pozitivnih i negativnih svojstvenih vrednosti

Dirakov i Vajlovi spinori

- \implies 2^n -dimenzionalni prostor \mathbb{R}^n se razlaže na dva potprostora dimenzije 2^{n-1} :

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \quad (19)$$

- ψ je Dirakov spinor, a
- ψ_+, ψ_- Vajlovi ili kiralni levi i desni spinor
- Reprezentacija Klifordove algebre dobijena na ovaj način naziva se Spinorska reprezentacija

Spinorska reprezentacija u neparnom broju dimenzija

- $N = 2n + 1$
- Potrebno je $2n + 1$ matrica: $\gamma_1, \dots, \gamma_{2n}, \gamma_{2n+1}$

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\mathbb{1}, \quad \mu, \nu = 1, \dots, 2n + 1. \quad (20)$$

- Već imamo $\gamma_1, \dots, \gamma_{2n}$ i $\gamma = \alpha\gamma_1 \cdots \gamma_{2n}$
- Proglasimo $\gamma_{2n+1} = \gamma$:

$$\gamma_{2n+1}^\dagger = \gamma_{2n+1}, \quad \gamma_{2n+1}^2 = \mathbb{1}, \quad \{\gamma_{2n+1}, \gamma_i\} = 0 \quad (21)$$

za neko $i = 1, \dots, 2n$ i opet imamo celu Klifordovu algebru

- Ali nemamo više matricu kiralnosti \Rightarrow nema više Vajlovih spinora
- $\gamma_1 \cdots \gamma_{2n}\gamma_{2n+1} = \alpha\gamma_1 \cdots \gamma_{2n}\gamma_1 \cdots \gamma_{2n} = \pm\eta\mathbb{1}, \eta \in \mathbb{C}$
- U neparnom broju dimenzija postoje dve neekvivalentne IR
- Izbor $-\gamma_{2n+1}$ ili γ_{2n+1}
- Različiti znaci $\pm\eta$ odgovaraju različitim reprezentacijama

Glatka mnogostruktost

- Realna mnogostruktost M dimenzije n ($\dim_{\mathbb{R}} M = n$) je Hausdorfov topološki prostor koji lokalno liči na \mathbb{R}^n
- To je skup otvornih podskupova U_i zajedno sa homeomorfizmima

$$U_i \xrightarrow{\phi_i} \mathbb{R}^n$$

koji se nazivaju **karte**. Skup karata koji pokriva celu mnogostruktost naziva se **atlas**

- Mnogostruktost M je **diferencijabilna** ili **glatka** ako na preseku karti:

$$\begin{array}{ccc} U_i \cap U_j & & \\ \phi_i^{-1} \nearrow & & \searrow \phi_j \\ \mathbb{R}^n & \xrightarrow{\phi_i^{-1} \circ \phi_j} & \mathbb{R}^n \end{array}$$

postoji glatka funkcija prelaza $g_{ij} = \phi_i^{-1} \circ \phi_j$ (**difeomorfizam**)

Diferencijalne forme

- Razmotrimo antisimetrične tenzore ranga p , $0 \leq p \leq n$
- Uvodimo diferencijal dx^μ , $\mu = 1, \dots, n$ na datoj lokalnoj karti
- Definišemo **kosi proizvod** kao antisimetrični tensorski proizvod:

$$dx^\mu \wedge dx^\nu = \frac{1}{2}(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu) = -dx^\nu \wedge dx^\mu \quad (22)$$

- Definišemo **diferencijalnu p -formu**:

$$\alpha_p = \frac{1}{p!} \alpha_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}. \quad (23)$$

- *Primeri:* 0-forma je obična skalarna funkcija: $\alpha_0 = \phi(x)$, 1-forma je kovarijantni vektor: $\alpha_1 = A = A_\mu dx^\mu$, 2-forma je antisimetrični tenzor ranga 2: $\alpha_2 = B = \frac{1}{2}B_{\mu\nu} dx^\mu \wedge dx^\nu$
- Skup svih p -formi: $\Omega^p(M)$
- Kosi proizvod p -forme α_p and q -forme β_q je preslikavanje:

$$\Omega^p(M) \times \Omega^q(M) \longrightarrow \Omega^{p+q}(M) : \quad \lambda_{p+q} = \alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p \quad (24)$$

Spoljašnji izvod

- Spoljašnji izvod je preslikavanje : $d : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$
- U koordinatnoj reprezentaciji $d = dx^\mu \partial_\mu \wedge$
- Primer (skalarno polje): $d\phi = \partial_\mu \phi dx^\mu = \text{grad}\phi$
- Primer (gradijentni potencijal): $dA = \partial_\mu dx^\mu \wedge A_\nu dx^\nu = \partial_\mu A_\nu dx^\mu \wedge dx^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu = \frac{1}{2}F_{\mu\nu} dx^\mu \wedge dx^\nu$
 $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- Spoljašnji izvod je rotor kada deluje na 1-formu u 3 dimenzije.
- Zadovoljava Lajbnicovo pravilo:

$$d(\alpha_p \wedge \beta_q) = (d\alpha_p) \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q. \quad (25)$$

- Uslov nilpotentnosti:

$$d^2 = 0. \quad (26)$$

Lako se pokazuje: $d^2\alpha_p = d(dx^\mu \partial_\mu \wedge \alpha_p) = dx^\nu \wedge dx^\mu \wedge \partial_\mu \partial_\nu \alpha_p = 0.$

Stoksova teorema i Hodžov dual

- Stoksova teorema: Za $\dim_{\mathbb{R}} M = n$ važi

$$\int_M d\alpha_{n-1} = \int_{\partial M} \alpha_n, \quad (27)$$

gde je ∂M granica mnogostruktosti M .

- Za datu Rimanovu mnogostruktost (M, g) uvodimo Hodžov $*$ -dual kao preslikavanje

$$*: \Omega^p(M) \longrightarrow \Omega^{n-p}(M)$$

definisano:

$$*dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = \frac{\sqrt{g}}{(n-p)!} \epsilon^{\mu_1 \cdots \mu_p}_{\mu_{p+1} \cdots \mu_n} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_n}. \quad (28)$$

Najviša forma–element zapremine

- Definišemo meru integracije:

$$\text{Vol} = *1 = \frac{\sqrt{g}}{n!} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} \quad (29)$$

$$= \sqrt{g} dx^1 \wedge \dots \wedge dx^n \equiv \sqrt{g} d^n x. \quad (30)$$

$$*\text{Vol} = 1, \quad \sqrt{g} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} = \epsilon^{\mu_1 \dots \mu_n} \text{Vol}. \quad (31)$$

- Uvodjenje Hodžovog $*$ -duala omogućava izražavanje divergencije vektora
- U ravnom prostoru sa euklidskom signaturom: $\sqrt{g} = 1$, ako je A 1-forma:

$$\begin{aligned} d * A &= d \left(\frac{1}{(n-1)!} A_\nu \epsilon^\nu_{\mu_2 \dots \mu_n} dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n} \right) \\ &= \frac{1}{(n-1)!} A_{\nu, \rho} \epsilon^\nu_{\mu_2 \dots \mu_n} dx^\rho \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n} \\ &= \frac{1}{(n-1)!} A_{\nu, \rho} (n-1)! g^{\nu \rho} \text{Vol} = \text{Vol} A^\mu_{,\mu} = \text{Vol} \operatorname{div} A. \end{aligned} \quad (32)$$

- Još jedan Hodžov dual daje:

$$*d * A = \operatorname{div} A. \quad (33)$$

Metrički unutrašnji proizvod

- Metrički unutrašnji proizvod na $\Omega^p(M, \mathbb{R})$:

$$(\alpha, \beta) = \int_M \alpha \wedge * \beta, \quad \alpha, \beta \in \Omega^p(M, \mathbb{R}) \quad (34)$$

- U komponentama:

$$\begin{aligned} \alpha \wedge * \beta &= \alpha \wedge \frac{\sqrt{g}}{p!(n-p)!} \beta_{\mu_1 \dots \mu_p} \epsilon^{\mu_1 \dots \mu_p}_{\mu_{p+1} \dots \mu_n} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_n} \\ &= \frac{\sqrt{g}}{p!^2(n-p)!} \alpha_{\mu_1 \dots \mu_p} \beta_{\nu_1 \dots \nu_p} \epsilon^{\nu_1 \dots \nu_p}_{\mu_{p+1} \dots \mu_n} \\ &\quad \times dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \wedge dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_n} \\ &= \frac{\sqrt{g}}{p!^2(n-p)!} \alpha_{\mu_1 \dots \mu_p} \beta_{\nu_1 \dots \nu_p} \epsilon^{\nu_1 \dots \nu_p}_{\mu_{p+1} \dots \mu_n} \epsilon^{\mu_1 \dots \mu_n} dx^1 \wedge \dots \wedge dx^n \\ &= \frac{\text{Vol}}{p!^2(n-p)!} \alpha_{\mu_1 \dots \mu_p} \beta_{\nu_1 \dots \nu_p} p!(n-p)! g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p} \\ &= \frac{\text{Vol}}{p!} \alpha^{\mu_1 \dots \mu_p} \beta_{\mu_1 \dots \mu_p}. \end{aligned} \quad (35)$$

- Integracijom unutrašnjeg metričkog proizvoda:

$$\int_M \alpha \wedge * \beta = \int_M \text{Vol} \frac{1}{p!} \alpha^{\mu_1 \dots \mu_p} \beta_{\mu_1 \dots \mu_p}. \quad (36)$$

- Kinetički član za realno skalarno polje:

$$S_\phi = \frac{1}{2} \int \sqrt{g} d^n x \partial_\mu \phi \partial^\mu \phi = \frac{1}{2} \int d\phi \wedge *d\phi. \quad (37)$$

- Kinetički član za Abelovo gradijentno polje jačine $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ koje je 2-forma:

$$\frac{1}{2} \int \sqrt{g} d^n x F_{\mu\nu} F^{\mu\nu} = \int F \wedge *F. \quad (38)$$

- Za kompleksno skalarno polje:

$$S_\phi = \int \sqrt{g} d^n x \partial_\mu \phi^* \partial^\mu \phi = \int d\phi^* \wedge *d\phi. \quad (39)$$

Lokalne teorije, kovariantni izvod

- Spinori su prirodno definisani u Lorencovom (euklidskom) prostoru
- Svaka mnogostruktost je lokalno ravna
- Spinorsko polje $\psi(x)$ se lokalno transformiše pri dejstvu *Spin* grupe:

$$\psi \longrightarrow g(x)\psi \quad (40)$$

- Invarijantnost lagranžijana \longrightarrow **kovariantni izvod**

$$D = d + \omega \quad (41)$$

- i **koneksija** $\omega(x)$ koja se transformiše kao:

$$\omega \longrightarrow g^{-1}(\omega + d)g. \quad (42)$$

- Zahtevamo da se $D\psi$ transformiše kao spinor:

$$\begin{aligned} D\psi &\longrightarrow (d + g\omega g^{-1} + gdg^{-1})g\psi \\ &= (dg)\psi + gd\psi + g\omega g^{-1}\psi + g(dg^{-1})g\psi \\ &= dg\psi + gd\psi + g\omega\psi - gg^{-1}dg\psi \\ &= g(d + \omega)\psi = gD\psi. \end{aligned} \quad (43)$$

Ortonormalni bazis u tangentnom prostoru

- U koordinatnom bazisu $\{\partial_\mu\}$ na tangentnom prostoru $T_p(M)$ mnogostrukosti M u tački p
- Korišćenjem **inverzne tetrade** e_α^μ konstruišemo **ortonormalni bazis** $\{e_\alpha\}$:

$$e_\alpha(x) = e_\alpha^\mu(x)\partial_\mu, \quad \alpha = 1, \dots, \dim_{\mathbb{R}} M. \quad (44)$$

- Ortonormalnost:

$$e_\alpha^\mu(x)e_\beta^\nu(x)g_{\mu\nu} = \eta_{\alpha\beta}. \quad (45)$$

- Svakom vektoru $v \in T_p(M)$ odgovara **dualni vektor (kovektor)** koji je functional:

$$\omega : T_p(M) \longrightarrow \mathbb{R}, \quad \omega(v) = \langle \omega | v \rangle \in \mathbb{R}. \quad (46)$$

- Kovektori obrazuju kotangentni prostor $T_p^*(M)$
- U koordinatnoj reprezentaciji $\{dx^\mu\}$ je bazis kotangentnog prostora:

$$\langle dx^\mu | \partial_\nu \rangle = \partial_\nu(dx^\mu) = \delta_\nu^\mu. \quad (47)$$

- **Bazisni vektori kotangentnog prostora su 1-forme**

Tetrada i metrika

- Korišćenjem tetrade definišemo **ortonormalni bazis** 1-formi:

$$\theta^\alpha(x) = \theta_\mu^\alpha(x) dx^\mu. \quad (48)$$

- Ortogonalnost:

$$\theta_\mu^\alpha(x) \theta_\nu^\beta(x) g^{\mu\nu}(x) = \eta^{\alpha\beta}. \quad (49)$$

- $dx^\mu = e_\alpha^\mu \theta^\alpha \implies$ metrički tenzor:

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{\alpha\beta} \theta^\alpha \otimes \theta^\beta. \quad (50)$$

- Provera konzistentnosti:

$$\langle \theta^\alpha | e_\beta \rangle = \langle \theta_\mu^\alpha dx^\mu | e_\beta^\nu \partial_\nu \rangle = \theta_\mu^\alpha e_\beta^\nu \langle dx^\mu | \partial_\nu \rangle = \theta_\mu^\alpha e_\beta^\mu = \delta_\beta^\alpha \quad (51)$$

- Tetrade θ_μ^α i e_α^μ su inverzi:

$$\theta_\mu^\alpha e_\alpha^\nu = \delta_\mu^\nu, \quad \theta_\mu^\alpha e_\beta^\mu = \delta_\beta^\alpha \quad (52)$$

- Lokalnom Lorencovom metrikom $\eta_{\alpha\beta}$ i inverzom $\eta^{\alpha\beta}$ podižemo i spuštamo indekse

Spinska koneksija

- Izbor ortonormalnog bazisa je jedinstven do na lokalne Lorencove transformacije:

$$\theta^\alpha(x) \longrightarrow \theta'^\alpha(x) = \Lambda^{-1}{}^\alpha_\beta(x) \theta^\beta(x). \quad (53)$$

- Lokalna simetrija \longrightarrow gradijentno polje, 1-forma sa transformacijom:

$$\omega \longrightarrow \omega' = \Lambda^{-1}(\omega + d)\Lambda. \quad (54)$$

- 1-forma koneksije se naziva **spinska koneksija**
- Dekorisano indeksima:

$$\omega^\alpha{}_\beta \longrightarrow \omega'^\alpha{}_\beta = \Lambda^{-1}{}^\alpha{}_\gamma \omega^\gamma{}_\delta \Lambda^\delta{}_\beta + \Lambda^{-1}{}^\alpha{}_\gamma d\Lambda^\gamma{}_\beta. \quad (55)$$

- Kao 1-forma može da se razvije u ortonormalnom bazisu:

$$\omega^\alpha{}_\beta = \omega^\alpha{}_{\mu\beta} dx^\mu = \omega^\alpha{}_{\gamma\beta} \theta^\gamma. \quad (56)$$

- Kovarijantni izvod za Lorencovu grupu:

$$D = d + \frac{1}{2} \omega_{\alpha\beta} \Sigma^{\alpha\beta}, \quad (57)$$

- $\Sigma^{\alpha\beta} = \frac{1}{4} [\gamma^\alpha, \gamma^\beta]$ su Lorencovi generatori

Strukturne jednačine

- Elementi koordinatnog bazisa medjusobno komutiraju $[\partial_\mu, \partial_\nu] = 0$
- Elementi ortonormalnog bazisa, ne nužno:

$$[e_\alpha, e_\beta] = C^\gamma{}_{\alpha\beta} e_\gamma. \quad (58)$$

- $C^\gamma{}_{\alpha\beta}$ su strukturne konstante ili objekti anholonomije

$$[e_\alpha, e_\beta] = e_\alpha e_\beta^\mu \partial_\mu - e_\beta e_\alpha^\mu \partial_\mu = (e_\alpha e_\beta^\mu - e_\beta e_\alpha^\mu) \theta_\mu^\gamma e_\gamma. \quad (59)$$

$$C^\gamma{}_{\alpha\beta} = \theta_\mu^\gamma (e_\alpha e_\beta^\mu - e_\beta e_\alpha^\mu). \quad (60)$$

- Diferencijal tetrade:

$$\begin{aligned} d\theta^\gamma &= d\theta_\mu^\gamma \wedge dx^\mu = (e_\alpha \theta_\mu^\gamma) \theta^\alpha \wedge e_\beta^\mu \theta^\beta = -\frac{1}{2} \theta_\mu^\gamma (e_\alpha e_\beta^\mu - e_\beta e_\alpha^\mu) \theta^\alpha \wedge \theta^\beta \\ &= -\frac{1}{2} C^\gamma{}_{\alpha\beta} \theta^\alpha \wedge \theta^\beta. \end{aligned} \quad (61)$$

- Strukturne jednačine:

$$d\theta^\alpha = -\frac{1}{2} C^\alpha{}_{\beta\gamma} \theta^\beta \wedge \theta^\gamma. \quad (62)$$

1. Kartanova jednačina strukture–torzija

- Spoljašnji izvod tetradne 1-forme je 2-forma:

$$d\theta^\alpha = \partial_\mu dx^\mu \wedge \theta_\nu^\alpha dx^\nu = \frac{1}{2}(\partial_\mu \theta_\nu^\alpha - \partial_\nu \theta_\mu^\alpha) dx^\mu \wedge dx^\nu. \quad (63)$$

- Transformiše se kao 2-forma, ali ne i kao Lorencov vektor

$$d\theta' = d(\Lambda^{-1}\theta) = \Lambda^{-1}d\theta + (d\Lambda^{-1})\theta \quad (64)$$

- Uvedemo član $\omega \wedge \theta$ sadrži spinsku koneksiju:

$$\omega' = \Lambda^{-1}d\Lambda + \Lambda^{-1}\omega\Lambda \quad (65)$$

- Transformacija novog izraza:

$$\begin{aligned} d\theta' + \omega' \wedge \theta' &= \Lambda^{-1}d\theta + d\Lambda^{-1} \wedge \theta + (\Lambda^{-1}d\Lambda + \Lambda^{-1}\omega\Lambda) \wedge \Lambda^{-1}\theta \\ &= \Lambda^{-1}d\theta + d\Lambda^{-1} \wedge \theta - \Lambda^{-1}\Lambda \wedge d\Lambda^{-1} \wedge \theta + \Lambda^{-1}\omega \wedge \theta \\ &= \Lambda^{-1}(d\theta + \omega \wedge \theta). \end{aligned} \quad (66)$$

- Definišemo **1-formu torzije**:

$$T^\alpha = d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta. \quad (67)$$

- Prva Kartanova jednačina strukture

2. Kartanova jednačina strukture–krivina

- Komutator dva kovarijantna izvoda:

$$\begin{aligned} D \wedge D &= D_\alpha \theta^\alpha \wedge D_\beta \theta^\beta = D_\mu dx^\mu \wedge D_\nu dx^\nu \\ &= \frac{1}{2}(D_\alpha D_\beta - D_\beta D_\alpha)\theta^\alpha \theta^\beta = \frac{1}{2}(D_\mu D_\nu - D_\nu D_\mu)dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}R_{\alpha\beta}\theta^\alpha \wedge \theta^\beta = \frac{1}{2}R_{\mu\nu}dx^\mu \wedge dx^\nu \end{aligned} \quad (68)$$

- Prepoznajemo Rimanov tenzor u ortogonalnom i koordinatnom bazisu
- Ova 2-forma je krivina spinske koneksije—**2-forma krivine**:

$$\Omega = D \wedge D = (d + \omega) \wedge (d + \omega) = d\omega + \omega \wedge \omega \quad (69)$$

- U komponentama:

$$\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta. \quad (70)$$

- Ovo je druga **Kartanova jednačina strukture**
- U ortogonalnom i koordinatnom bazisu:

$$\Omega^\alpha{}_\beta = \frac{1}{2}R^\alpha{}_{\beta\gamma\delta}\theta^\gamma \wedge \theta^\delta = \frac{1}{2}R^\alpha{}_{\beta\mu\nu}dx^\mu \wedge dx^\nu, \quad (71)$$

$$R^\alpha{}_{\beta\gamma\delta} = e^\mu{}_\gamma e^\nu{}_\delta R^\alpha{}_{\beta\mu\nu}, \quad R^\alpha{}_{\beta\mu\nu} = \theta^\gamma_\mu \theta^\delta_\nu R^\alpha{}_{\beta\gamma\delta}. \quad (72)$$

Kartanove strukturne jednačine

- Kartanove strukturne jednačine u matričnoj formi:

$$T = d\theta + \omega \wedge \theta, \quad \Omega = d\omega + \omega \wedge \omega. \quad (73)$$

- Delovanjem spoljsnjim izvodom:

$$\begin{aligned} dT &= d\omega \wedge \theta - \omega \wedge d\theta = d\omega \wedge \theta - \omega \wedge (T - \omega \wedge \theta) \\ &= d\omega \wedge \theta - \omega \wedge T + \omega \wedge \omega \wedge \theta = \Omega \wedge \theta - \omega \wedge T, \end{aligned} \quad (74)$$

$$\begin{aligned} d\Omega &= d\omega \wedge \omega - \omega \wedge d\omega = (\Omega - \omega \wedge \omega) \wedge \omega - \omega \wedge (\Omega - \omega \wedge \omega) \\ &= \Omega \wedge \omega - \omega \wedge \Omega, \end{aligned} \quad (75)$$

- Bjankijevi identiteti:

$$dT + \omega \wedge T = \Omega \wedge \theta, \quad d\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0. \quad (76)$$

- Kartanove strukturne jednačine i tetradi formalizam omogućavaju kuplovanje fermiona sa gravitacijom, što u slučaju Rimanove geometrije nije moguće

Odsečena Hajzenbergova algebra

- Odsečena Hajzenbergova algebra je matrična aproksimacija Hajzenbergove algebre
- Dobija se odsecanjem beskonačnih matrica koje reprezentuju koordinatu i impuls linearog harmonijskog oscilatora
- Zadata je komutacionim relacijama:

$$[\mu x, \mu y] = i\epsilon(1 - z), \quad (77)$$

$$[\mu x, \mu z] = i\epsilon\mu(yz + zy),$$

$$[\mu y, \mu z] = -i\epsilon\mu(xz + zx).$$

- U nekomutativnoj geometriji, moguće je identifikovati fazni prostor sa pozicionim prostorom
- U slučaju unutrašnjih izvoda e_α , za date *impulse* $p_\alpha \in \mathcal{A}$:

$$e_\alpha f = [p_\alpha, f] \quad (78)$$

$$e_\alpha^\mu = [p_\alpha, x^\mu] \quad (79)$$

- Identifikujemo impulse:

$$\epsilon p_1 = i\mu^2 y, \quad \epsilon p_2 = -i\mu^2 x, \quad \epsilon p_3 = i\mu(\mu z - \frac{1}{2}) \quad (80)$$

- Algebra impulsa [Madore]:

$$[p_1, p_2] = \frac{\mu^2}{2i} + \mu p_3, \quad (81)$$

$$[p_2, p_3] = \mu p_1 - i\epsilon(p_1 p_3 + p_3 p_1),$$

$$[p_3, p_1] = \mu p_2 - i\epsilon(p_2 p_3 + p_3 p_2).$$

- Uopštene komutacione relacije za p uz uslov $d^2 = 0$:

$$2P^{\gamma\delta}_{\alpha\beta}p_\gamma p_\delta - F^\gamma_{\alpha\beta}p_\gamma - \frac{1}{i\epsilon}K_{\alpha\beta} = 0 \quad (82)$$

- Izborom koneksije ($\mu = \epsilon = 1$)[Buric, Madore, Grosse, Wohlgenannt]:

$$\omega_{12} = -\omega_{21} = \left(-\frac{1}{2} + 2ip_3\right)\theta^3 = \left(\frac{1}{2} - 2z\right)\theta^3, \quad (83)$$

$$\omega_{13} = -\omega_{31} = \frac{1}{2}\theta^2 + 2ip_2\theta^3 = \frac{1}{2}\theta^2 + 2x\theta^3,$$

$$\omega_{23} = -\omega_{32} = -\frac{1}{2}\theta^1 - 2ip_1\theta^3 = -\frac{1}{2}\theta^1 + 2y\theta^3.$$

- U komponentama

$$\omega_{132} = -\omega_{231} = \frac{1}{2} - 2z, \quad (84)$$

$$\omega_{123} = -\omega_{321} = \frac{1}{2}, \quad \omega_{133} = -\omega_{331} = 2x,$$

$$\omega_{213} = -\omega_{312} = -\frac{1}{2}, \quad \omega_{233} = -\omega_{332} = 2y.$$

Ortonormalni bazis i 3-forma zapremine

- Komutacione relacije za ortogonalni bazis:

$$(\theta^1)^2 = 0, \quad (\theta^2)^2 = 0, \quad (\theta^3)^2 = 0, \quad \{\theta^1, \theta^2\} = 0 \quad (85)$$

$$\{\theta^1, \theta^3\} = i\epsilon(\theta^2\theta^3 - \theta^3\theta^2), \quad \{\theta^2, \theta^3\} = i\epsilon(\theta^3\theta^1 - \theta^1\theta^3) \quad (86)$$

- Ekstenzijom na algebru 3-formi:

$$\theta^1\theta^3\theta^1 = \theta^2\theta^3\theta^2, \quad \theta^3\theta^1\theta^3 = \theta^3\theta^1\theta^3 = 0 \quad (87)$$

$$\theta^1\theta^2\theta^3 = -\theta^2\theta^1\theta^3 = \theta^3\theta^1\theta^2 = -\theta^3\theta^2\theta^1 = i\frac{\epsilon^2 - 1}{2\epsilon}\theta^2\theta^3\theta^2 \quad (88)$$

$$\theta^1\theta^3\theta^2 = -\theta^2\theta^3\theta^1 = i\frac{\epsilon^2 + 1}{2\epsilon}\theta^2\theta^3\theta^2 \quad (89)$$

- Postoji samo jedna linearno nezavisna forma zapremine:

$$\Theta = -\frac{i}{2\epsilon}\theta^2\theta^3\theta^2 \quad (90)$$

Dirakove matrice u dve i tri dimenzije

- U dve i tri dimenzije spinorska reprezentacija je dvodimenzionalna
- Prirodni izbor Dirakovih matrica su Paulijeve matrice:

$$\gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (91)$$

- Ermitske matrice koje zadovoljavaju:

$$\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2. \quad (92)$$

- Matrica kiralnosti u dve dimenzije:

$$\gamma_3 = -i\gamma_1\gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \quad (93)$$

- Ove tri matrice formiraju Klifordovu algebru u 3 dimenzije

Dirakov spinor u 3 dimenzije

- Za Klifordovu algebru u 3 dimenzije

$$\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3. \quad (94)$$

- Postoje dve neekvivalentne ireducibilne reprezentacije. Biramo:

$$\gamma_4 = -i\gamma_1\gamma_2\gamma_3 = 1 \quad (95)$$

- Pri $Spin(3)$ transformacijama, Dirakov spinor se transformiše:

$$\delta\psi = \frac{1}{4}\omega^{\alpha\beta}\gamma_{\alpha\beta}\psi, \quad (96)$$

gde je $\gamma_{\alpha\beta} = \gamma_\alpha\gamma_\beta$ za $\alpha \neq \beta$, a nula inače

- Zahtevamo da Dirakov konjugovani spinor $\bar{\psi}$ poništi ovu transformaciju formirajući skalar $\bar{\psi}\psi$

$$\delta\psi^\dagger = \psi^\dagger \left(\frac{1}{4}\omega^{\alpha\beta}\gamma_{\alpha\beta}^\dagger \right) = \psi^\dagger \left(\frac{1}{4}\omega^{\alpha\beta}\gamma_{\beta\alpha} \right) = \psi^\dagger \left(-\frac{1}{4}\omega^{\alpha\beta}\gamma_{\alpha\beta} \right) \quad (97)$$

- $\implies \bar{\psi} = \psi^\dagger$

- Fermionsko dejstvo u tri dimenzije u izraženo preko formi:

$$S_{Dirac} = - \int \text{Tr}[(D\psi)\bar{\psi} - \psi D\bar{\psi}] \wedge V \wedge V \wedge \gamma_4. \quad (98)$$

- Postoji torzija koju neminimalno kuplujemo sa fermionima:

$$S_{tor} = \int \text{Tr}\psi\bar{\psi}T^\alpha\gamma_\alpha V. \quad (99)$$

$$\begin{aligned} T^1 &= \mu \left(\frac{1}{2} - \mu z \right) (\theta^2\theta^3 + \theta^3\theta^2) \\ T^2 &= -\mu \left(\frac{1}{2} - \mu z \right) (\theta^1\theta^3 + \theta^3\theta^1) \\ T^3 &= -\mu^2 x(\theta^1\theta^3 + \theta^3\theta^1) - \mu^2 y(\theta^2\theta^3 + \theta^3\theta^2). \end{aligned} \quad (100)$$

Dirakovo dejstvo

- Raspisano preko komponenata:

$$\begin{aligned} S_{Dirac} &= - \int \text{Tr}[(D\psi)\bar{\psi} - \psi D\bar{\psi}] \wedge V \wedge V \wedge \gamma_4 \\ &= - \int \text{Tr}[(D_\alpha \psi)\bar{\psi} - \psi D_\alpha \bar{\psi}] \theta^\alpha \theta^\beta \gamma_\beta \theta^\gamma \gamma_\gamma \\ &= - \int \text{Tr} X_\alpha \gamma_\beta \gamma_\gamma \theta^\alpha \theta^\beta \theta^\gamma, \end{aligned} \tag{101}$$

gde je:

$$X_\alpha = (D_\alpha \psi)\bar{\psi} - \psi D_\alpha \bar{\psi}. \tag{102}$$

Sumiranjem komponenti:

$$\begin{aligned} S_{Dirac} &= - \int \text{Tr} X_\alpha \gamma_\beta \gamma_\gamma \theta^\alpha \theta^\beta \theta^\gamma \\ &= - \int \text{Tr}(X_1 \gamma_3 \gamma_1 \theta^1 \theta^3 \theta^1 + X_1 \gamma_2 \gamma_3 \theta^1 \theta^2 \theta^3 + X_1 \gamma_3 \gamma_2 \theta^1 \theta^3 \theta^2 \\ &\quad + X_2 \gamma_3 \gamma_2 \theta^2 \theta^3 \theta^2 + X_2 \gamma_1 \gamma_3 \theta^2 \theta^1 \theta^3 + X_2 \gamma_3 \gamma_1 \theta^2 \theta^3 \theta^1 \\ &\quad + X_3 \gamma_1 \gamma_2 \theta^3 \theta^1 \theta^2 + X_3 \gamma_2 \gamma_1 \theta^3 \theta^2 \theta^1). \end{aligned} \tag{103}$$

Dirakovo dejstvo 2

- Izražavanjem preko jedinstvene forme zapremine $\theta^2\theta^3\theta^2$:

$$\begin{aligned} S_{Dirac} = & -i \int \text{Tr} \left(X_1 \gamma_2 + X_1 \gamma_1 i \frac{\epsilon^2 - 1}{2\epsilon} - X_1 \gamma_1 i \frac{\epsilon^2 + 1}{2\epsilon} \right. \\ & - X_2 \gamma_1 + X_2 \gamma_2 i \frac{\epsilon^2 - 1}{2\epsilon} - X_2 \gamma_2 i \frac{\epsilon^2 + 1}{2\epsilon} \\ & \left. + X_3 \gamma_3 i \frac{\epsilon^2 - 1}{2\epsilon} + X_3 \gamma_3 i \frac{\epsilon^2 + 1}{2\epsilon} \right) \theta^2 \theta^3 \theta^2. \end{aligned} \quad (104)$$

$$\Theta = -\frac{i}{2\epsilon} \theta^2 \theta^3 \theta^2:$$

$$S_{Dirac} = 2\epsilon \int \text{Tr} \left[X_1 \left(\gamma_2 - \frac{i}{\epsilon} \gamma_1 \right) - X_2 \left(\gamma_1 + \frac{i}{\epsilon} \gamma_2 \right) + X_3 \gamma_3 \left(\epsilon - \frac{1}{\epsilon} \right) \right] \Theta. \quad (105)$$

Kovarijantni izvodi

$$\begin{aligned} S_{Dirac} &= -\frac{1}{2} \int \text{Tr}[X_1(-\epsilon\gamma_2 + i\gamma_1) + X_2(\epsilon\gamma_1 + i\gamma_2) + iX_3\gamma_3(1 - \epsilon^2)]\Theta \\ &= S_{D1} + S_{D2} + S_{D3}. \end{aligned} \quad (106)$$

$$\begin{aligned} D_1\psi &= \left(\mathbf{e}_1 - \frac{i}{4}\mu\gamma^1\right)\psi, & D_1\bar{\psi} &= \mathbf{e}_1\bar{\psi} + \frac{i}{4}\mu\bar{\psi}\gamma^1 \\ D_2\psi &= \left(\mathbf{e}_2 - \frac{i}{4}\mu\gamma^2\right)\psi, & D_2\bar{\psi} &= \mathbf{e}_2\bar{\psi} + \frac{i}{4}\mu\bar{\psi}\gamma^2 \\ D_3\psi &= \left[\mathbf{e}_3 + \frac{i}{4}\mu\gamma^3 - i\mu^2(x\gamma^2 - y\gamma^1 + z\gamma^3)\right]\psi, \\ D_3\bar{\psi} &= \mathbf{e}_3\bar{\psi} - \bar{\psi}\left[\frac{i}{4}\mu\gamma^3 + i\mu^2(x\gamma^2 - y\gamma^1 + z\gamma^3)\right] \end{aligned} \quad (107)$$

Primer računa

$$\begin{aligned} S_{D1} &= -\frac{1}{2} \int \text{Tr}[X_1(-\epsilon\gamma_2 + i\gamma_1)]\Theta \\ &= -\frac{1}{2} \int \text{Tr}\left\{[(D_1\psi)\bar{\psi} - \psi D_1\bar{\psi}](-\epsilon\gamma_2 + i\gamma_1)\right\}\Theta \\ &= -\frac{1}{2} \int \text{Tr}\left\{\left[\left(\mathbf{e}_1 - \frac{i}{4}\mu\gamma_1\right)\psi\right]\bar{\psi} - \psi\mathbf{e}_1\bar{\psi} - \frac{i}{4}\mu\psi\bar{\psi}\gamma_1\right\}(-\epsilon\gamma_2 + i\gamma_1)\Theta \\ &= -\frac{1}{2} \int \text{Tr}\left[-\epsilon(\mathbf{e}_1\psi)\bar{\psi}\gamma_2 + \epsilon\psi\mathbf{e}_1\bar{\psi}\gamma_2 + i(\mathbf{e}_1\psi)\bar{\psi}\gamma_1 - i\psi\mathbf{e}_1\bar{\psi}\gamma_1 \right. \\ &\quad \left. + \frac{i}{4}\epsilon\mu\gamma_1\psi\bar{\psi}\gamma_2 + \frac{i}{4}\epsilon\mu\psi\bar{\psi}\gamma_1\gamma_2 + \frac{1}{4}\mu\gamma_1\psi\bar{\psi}\gamma_1 + \frac{1}{4}\mu\psi\bar{\psi}\gamma_1\gamma_1 \right]\Theta \\ &= -\frac{1}{2} \int \text{Tr}\left[i\bar{\psi}\gamma_1\mathbf{e}_1\psi - i(\mathbf{e}_1\bar{\psi})\gamma_1\psi + \frac{1}{2}\mu\psi\bar{\psi} - \epsilon\bar{\psi}\gamma_2\mathbf{e}_1\psi + \epsilon(\mathbf{e}_1\bar{\psi})\gamma_2\psi \right]\Theta. \end{aligned} \tag{108}$$

Dirakov i torzioni član

$$S_{Dirac} = -\frac{1}{2} \int \text{Tr} \left\{ i\bar{\psi}\gamma^\alpha e_\alpha \psi - i(e_\alpha \bar{\psi})\gamma^\alpha \psi + \frac{1}{2}\mu\bar{\psi}\psi + 2\mu^2 z\bar{\psi}\psi \right. \\ \left. + \epsilon \left[i\bar{\psi}\gamma^\alpha \gamma^3 e_\alpha \psi - i(e_\alpha \bar{\psi})\gamma^\alpha \gamma^3 \psi \right] \right. \\ \left. - \epsilon^2 \left[i\bar{\psi}\gamma^3 e_3 \psi - i(e_3 \bar{\psi})\gamma^3 \psi - \frac{1}{2}\mu\bar{\psi}\psi + 2\mu^2 z\bar{\psi}\psi \right] \right\} \Theta \quad (109)$$

za $\alpha = 1, 2$.

$$S_{tor} = -\frac{1}{2} \int \text{Tr} \{ \psi\bar{\psi} [\epsilon\mu(\gamma^3 - \mu x\gamma^2 + \mu y\gamma^1 - 2\mu z\gamma^3) - \epsilon^2\mu(1 - 2\mu z)] \} \Theta. \quad (110)$$

Rezultat

Ukupno dejstvo:

$$S = -\frac{1}{2} \int \text{Tr} \left\{ i\bar{\psi}\gamma^\alpha e_\alpha \psi - i(e_\alpha \bar{\psi})\gamma^\alpha \psi + \frac{1}{2}\mu\bar{\psi}\psi + 2\mu^2 z\bar{\psi}\psi \right. \\ \left. + \epsilon \left[i\bar{\psi}\gamma^\alpha\gamma^3 e_\alpha \psi - i(e_\alpha \bar{\psi})\gamma^\alpha\gamma^3 \psi + \mu\bar{\psi}(\gamma^3 - \mu x\gamma^2 + \mu y\gamma^1 - 2\mu z\gamma^3)\psi \right] \right. \\ \left. - \epsilon^2 \left[i\bar{\psi}\gamma^3 e_3 \psi - i(e_3 \bar{\psi})\gamma^3 \psi + \frac{1}{2}\mu\bar{\psi}\psi \right] \right\} \Theta \quad (111)$$

Posle dimenzijske redukcije $z = 0$, $e_3 = 0$ dobijamo dejstvo

$$S = -\frac{1}{2} \int \text{Tr} \left\{ i\bar{\psi}\gamma^\alpha e_\alpha \psi - i(e_\alpha \bar{\psi})\gamma^\alpha \psi + \frac{1}{2}\mu\bar{\psi}\psi \right. \\ \left. + \epsilon \left[i\bar{\psi}\gamma^\alpha\gamma^3 e_\alpha \psi - i(e_\alpha \bar{\psi})\gamma^\alpha\gamma^3 \psi + \mu\bar{\psi}(\gamma^3 - \mu x\gamma^2 + \mu y\gamma^1)\psi \right] - \frac{\epsilon^2}{2}\mu\bar{\psi}\psi \right\} \Theta \quad (112)$$

- Kvadrat kernela ovog dejstva u slučaju velike nekomutativnosti odgovara Melerovom kernelu koji je u primeru kalibracionih teorija omogućio dobro UV ponašanje
- Očekujemo da će i teorija sa fermionima imati dobro UV ponašanje