

# Uopštenja talasne jednačine u okviru teorije frakcionog računa

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# Talasna jednačina kao sistem PDJ

- Klasična talasna jednačina (u jednoj prostornoj dimenziji)

$$\frac{\partial^2}{\partial t^2} u(x, t) = c^2 \frac{\partial^2}{\partial x^2} u(x, t), \quad c = \sqrt{E/\rho},$$

dobijena je iz sistema tri PDJ:

- jednačina kretanja

$$\frac{\partial}{\partial x} \sigma(x, t) = \rho \frac{\partial^2}{\partial t^2} u(x, t),$$

$\sigma$  - napon,  $\rho$  - gustina i  $u$  - pomeranje,

- konstitutivna jednačina - elastično telo - Hukov zakon

$$\sigma(x, t) = E \varepsilon(x, t)$$

$E$  - Jungov modul i  $\varepsilon$  - mera deformacije,

- mera deformacije

$$\varepsilon(x, t) = \frac{\partial}{\partial x} u(x, t).$$

# Uopštenja konstitutivne jednačine i deformacije

- Ukoliko viskoelastično telo poseduje svojstvo uticaja istorije

$$\int_0^1 \phi_\sigma(\alpha) {}_0D_t^\alpha \sigma(x, t) d\alpha = E \int_0^1 \phi_\varepsilon(\alpha) {}_0D_t^\alpha \varepsilon(x, t) d\alpha,$$

$$\sigma(x, t) + a_0 {}_0D_t^\alpha \sigma(x, t) = E (\varepsilon(x, t) + b_0 {}_0D_t^\alpha \varepsilon(x, t)),$$

$$\sum_{j=0}^n a_j {}_0D_t^{\alpha_j} \sigma(x, t) = \sum_{j=0}^n b_j {}_0D_t^{\alpha_j} \varepsilon(x, t),$$

- Ukoliko telo poseduje svojstvo prostorne nelokalnosti ( $\alpha \in (1, 3)$ )

$$\sigma(x, t) - I_c^\alpha D_x^\alpha \sigma(x, t) = E \varepsilon(x, t).$$

- Nelokalnost se može uvesti uopštavajući meru deformacije

$$\varepsilon(x, t) = \mathcal{E}_x^\beta u(x, t).$$

# Frakcioni integral

- Počevši od Košijeve formule ( $g \in L^1_{loc}(\mathbb{R}_+)$ ,  $t > 0$ ,  $n \in \mathbb{N}$ )

$$J^n g(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} g(\tau) d\tau = \frac{t^{n-1}}{(n-1)!} * g(t),$$

- zamenom  $n \in \mathbb{N}$  sa  $\alpha \in \mathbb{R}_+$  dobija se frakcioni integral

$$J^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau) d\tau = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * g(t), \quad t > 0.$$

- Uvodimo  $f_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ ,  $t > 0$ .
- FI zadovoljava svojstvo polugrupe (konvolucija i  $f_\alpha$ )

$$J^\alpha(J^\beta u) = J^\beta(J^\alpha u) = J^{\alpha+\beta} u.$$

- Frakcioni izvodi se uvode se inspirisani:

$$D^n J^n g = g \quad \text{i} \quad J^n D^n g(t) = g(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} D^k g(0).$$

# Riman-Liuvilov frakcioni izvod

## Definicija

Neka je  $u \in L^1_{loc}(\mathbb{R}_+)$ , tada je Riman-Liuvilov frakcioni izvod reda  $\alpha \in (m-1, m)$ ,  $m \in \mathbb{N}$  definisan

$${}_{0}^{RL}\mathrm{D}_t^\alpha u = \mathrm{D}^m(f_{m-\alpha} * u) = \mathrm{D}^m\left(\frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)} * u(t)\right).$$

- U operatorskom obliku:  ${}_{0}^{RL}\mathrm{D}_t^\alpha u = \mathrm{D}^m \mathrm{J}^{m-\alpha} u$ .
- Ukoliko je konvolucija eksplicitno zapisana

$${}_{0}^{RL}\mathrm{D}_t^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t \frac{u(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, \quad t > 0,$$

$${}_{0}^{RL}\mathrm{D}_t^\alpha (\mathrm{J}^\alpha u(t)) = u(t).$$

# Kaputov frakcioni izvod

## Definicija

Neka je  $u \in AC^m(\mathbb{R}_+)$ , tada je Kaputov frakcioni izvod reda  $\alpha \in (m-1, m)$ ,  $m \in \mathbb{N}$  definisan

$${}_0D_t^\alpha u = f_{m-\alpha} * D^m u = \frac{t^{m-1-\alpha}}{\Gamma(m-\alpha)} * D^m u.$$

- U operatorskom obliku:  ${}_0D_t^\alpha u = J^{m-\alpha} D^m u$ .
- Ukoliko je konvolucija eksplicitno zapisana

$${}_0D_t^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{D^m u(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \quad t > 0,$$

$$J^\alpha ({}_0D_t^\alpha u(t)) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} D^k u(0).$$

# Frakcioni izvod raspodeljenog reda

- Neka je  $u \in L^1_{loc}(0, \infty)$ ,  $\phi(\alpha) \geq 0$ ,  $\alpha \in [0, 1]$ ,  $\phi(0) = 0$ ,  $\phi'(\alpha) < 0$  i  $\phi \in C([0, 1])$ , tada je

$$D_\phi u(t) = \int_0^1 \phi(\alpha) {}_0D_t^\alpha u(t) d\alpha, \quad t > 0,$$

Riman-Liouvilov frakcioni izvod raspodeljenog reda.

- Laplasova transformacija (ukoliko je  $u_0 = 0$ )

$$\mathcal{L}[D_\phi u(t)](s) = \tilde{u}(s) \int_0^1 \phi(\alpha) s^\alpha d\alpha, \quad \operatorname{Re} s > 0.$$

- Za primenu - najjednostavniji izbor težinske funkcije  $\phi$

$$\phi(\alpha) = k^\alpha, \quad k > 0, \quad \alpha \in (0, 1),$$

$k$  je konstanta i  $[k] = s$ .

- Ostvarena dimenziona homogenost, jer je  $[{}_0D_t^\alpha u] = \frac{[u]}{s^\alpha}$ , te je  $[k^\alpha {}_0D_t^\alpha u] = [u]$ .

# Risov frakcioni izvod - I

- Risov frakcioni izvod može biti Kaputovog tipa. Tada je definisan ( $\beta \in (0, 1)$ ,  $x \in \mathbb{R}$ )

$$\begin{aligned}\mathcal{E}_x^\beta u(x) &= \frac{1}{2} \left( D_+^\beta u(x) - D_-^\beta u(x) \right) \\ &= \frac{1}{2} \frac{1}{\Gamma(1-\beta)} |x|^{-\beta} * \frac{d}{dx} u(x),\end{aligned}$$

- jer je

$$\begin{aligned}D_+^\beta u(x) &= \frac{1}{\Gamma(1-\beta)} \int_{-\infty}^x \frac{\frac{d}{d\zeta} u(\zeta)}{(x-\zeta)^\beta} d\zeta, \\ D_-^\beta u(x) &= -\frac{1}{\Gamma(1-\beta)} \int_x^\infty \frac{\frac{d}{d\zeta} u(\zeta)}{(\zeta-x)^\beta} d\zeta.\end{aligned}$$

- Risov frakcioni izvod uopštava prvi izvod, ali ne i nulti.

## Risov frakcioni izvod - II

- Risov frakcioni izvod Kaputovog tipa ( $\alpha \in (1, 2)$ ,  $x \in \mathbb{R}$ )

$$D_x^\alpha u = \frac{1}{2} (D_+^\alpha + D_-^\alpha) u = \frac{1}{2\Gamma(2-\alpha)} \frac{1}{|x|^{\alpha-1}} * \frac{d^2}{dx^2} u(x).$$

- Risov frakcioni izvod Kaputovog tipa ( $\alpha \in (2, 3)$ ,  $x \in \mathbb{R}$ )

$$D_x^\alpha u = \frac{1}{2} (D_+^\alpha + D_-^\alpha) u = \frac{1}{2\Gamma(3-\alpha)} \frac{\operatorname{sgn} x}{|x|^{\alpha-2}} * \frac{d^3}{dx^3} u(x).$$

- Oba uopštavaju drugi izvod.
- Definicije su ( $\alpha \in (n-1, n)$ ):

$$D_+^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^x \frac{\frac{d^n}{d\zeta^n} u(\zeta)}{(x-\zeta)^{\alpha-n+1}} d\zeta,$$

$$D_-^\alpha u(x) = (-1)^n \frac{1}{\Gamma(n-\beta)} \int_x^\infty \frac{\frac{d^n}{d\zeta^n} u(\zeta)}{(\zeta-x)^{\alpha-n+1}} d\zeta.$$

# Distributional fractional derivative I

- Distributional fractional derivative - defined as an inverse operator of  $J^\alpha$ .
- Family of tempered distributions supported by  $[0, \infty)$

$$f_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} H(t) & t \in \mathbb{R}, \alpha > 0, \\ \frac{d^n}{dt^n} \left[ \frac{t^{\alpha-1}}{\Gamma(\alpha)} H(t) \right] & t \in \mathbb{R}, n \in \mathbb{N}, \alpha + n > 0. \end{cases}$$

- Semigroup property

$$f_\alpha * f_\beta = f_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{R}.$$

# Distributional fractional derivative II

## Definition

Let  $h \in S'_+(\mathbb{R})$ , then the distributional fractional derivative of order  $\alpha \in (m-1, m)$ ,  $m \in \mathbb{N}$  is defined as

$$D_t^\alpha h = f_{m-\alpha} * D^m h = D^m [f_{m-\alpha} * h].$$

- Left, respectively right, inverse of  $J^\alpha$

$$D_t^\alpha J^\alpha h = D^m [f_{m-\alpha} * (f_\alpha * h)] = D^m [f_m * h] = D^m f_m * h = h,$$

$$J^\alpha D_t^\alpha h = f_\alpha * [f_{m-\alpha} * D^m h] = f_m * D^m h = D^m f_m * h = h.$$

# Distributed-order fractional derivative of a distribution

Let  $h \in \mathcal{S}'_+(\mathbb{R})$ ,  $\varphi \in \mathcal{S}(\mathbb{R})$ . Then the mappings

$$\alpha \mapsto D_t^\alpha h : \mathbb{R} \mapsto \mathcal{S}'_+(\mathbb{R}) \quad \text{and} \quad \alpha \mapsto \langle D_t^\alpha h(t), \varphi(t) \rangle : \mathbb{R} \mapsto \mathbb{R}$$

are smooth.

## Definition

Let  $\phi \in \mathcal{E}'(\mathbb{R})$ ,  $\text{supp } \phi \subset [0, 2]$  and  $h \in \mathcal{S}'_+(\mathbb{R})$ . Then distributed-order fractional derivative of  $h$

$$D_\phi h = \int_{\text{supp } \phi} \phi(\alpha) D_t^\alpha h d\alpha,$$

is defined as element of  $\mathcal{S}'_+(\mathbb{R})$  by

$$\langle D_\phi h(t), \varphi(t) \rangle = \langle \phi(\alpha), \langle D_t^\alpha h(t), \varphi(t) \rangle \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

# Frakciona talasna jednačina Cenerovog tipa

Frakciona talasna jednačina Cenerovog tipa predstavljena je sistemom uz početne i granične uslove ( $x \in \mathbb{R}$ ,  $t > 0$ )

$$\frac{\partial}{\partial x} \sigma(x, t) = \frac{\partial^2}{\partial t^2} u(x, t), \quad \varepsilon(x, t) = \frac{\partial}{\partial x} u(x, t),$$

$$\sigma(x, t) + \tau_0 D_t^\alpha \sigma(x, t) = \varepsilon(x, t) + {}_0 D_t^\alpha \varepsilon(x, t), \quad \tau < 1,$$

$$u(x, 0) = u_0(x), \quad \frac{\partial}{\partial t} u(x, 0) = v_0(x), \quad \sigma(x, 0) = 0, \quad \varepsilon(x, 0) = 0,$$

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} \sigma(x, t) = 0.$$

U distribucionoj postavci je oblika

$$\frac{\partial^2}{\partial t^2} u(x, t) = \mathcal{L}^{-1} \left[ \frac{1 + s^\alpha}{1 + \tau s^\alpha} \right] *_t \frac{\partial^2}{\partial x^2} u(x, t) + u_0(x) \delta'(t) + v_0(x) \delta(t).$$

## Teorema

Neka su  $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$ . Tada postoji jedinstveno rešenje  $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$  frakcione talasne jednačine Cenerovog tipa

$$u(x, t) = S(x, t) *_{x,t} (u_0(x)\delta'(t) + v_0(x)\delta(t)),$$

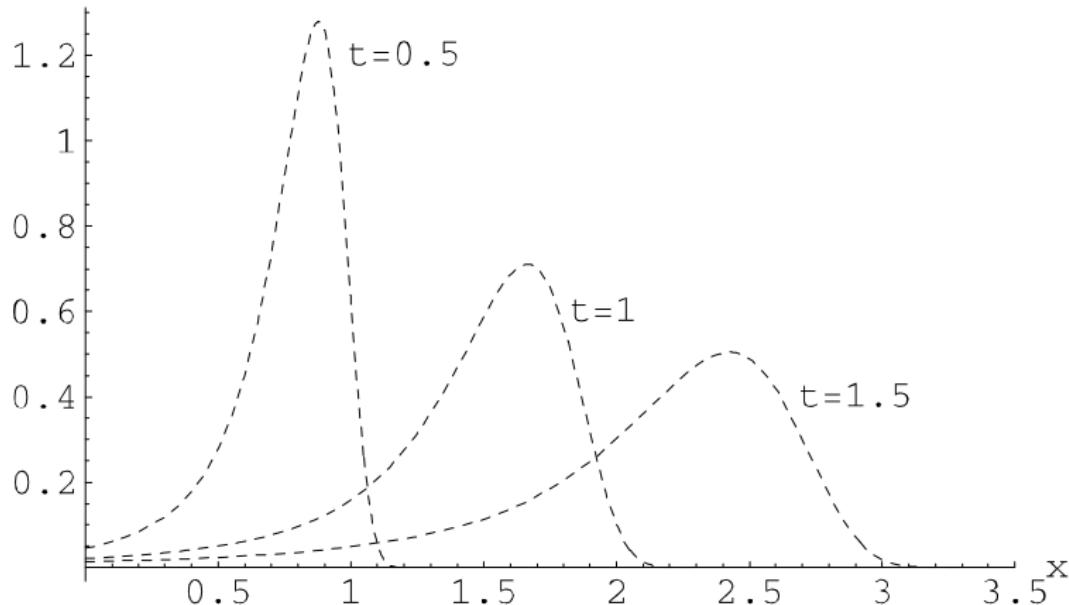
gde je

$$\begin{aligned} S(x, t) = & \frac{1}{2} + \frac{1}{4\pi i} \int_0^\infty \left( \sqrt{\frac{1 + \tau q^\alpha e^{i\alpha\pi}}{1 + q^\alpha e^{i\alpha\pi}}} e^{|x|q\sqrt{\frac{1 + \tau q^\alpha e^{i\alpha\pi}}{1 + q^\alpha e^{i\alpha\pi}}}} \right. \\ & \left. - \sqrt{\frac{1 + \tau q^\alpha e^{-i\alpha\pi}}{1 + q^\alpha e^{-i\alpha\pi}}} e^{|x|q\sqrt{\frac{1 + \tau q^\alpha e^{-i\alpha\pi}}{1 + q^\alpha e^{-i\alpha\pi}}}} \right) \frac{e^{-qt}}{q} dq, \end{aligned}$$

fundamentalno rešenje,  $S \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$  sa nosačem u konusu  $|x| < \frac{t}{\sqrt{\tau}}$ .

# Grafik rešenja $u$ za $u_0 = \delta$ i $v_0 = 0$ - I

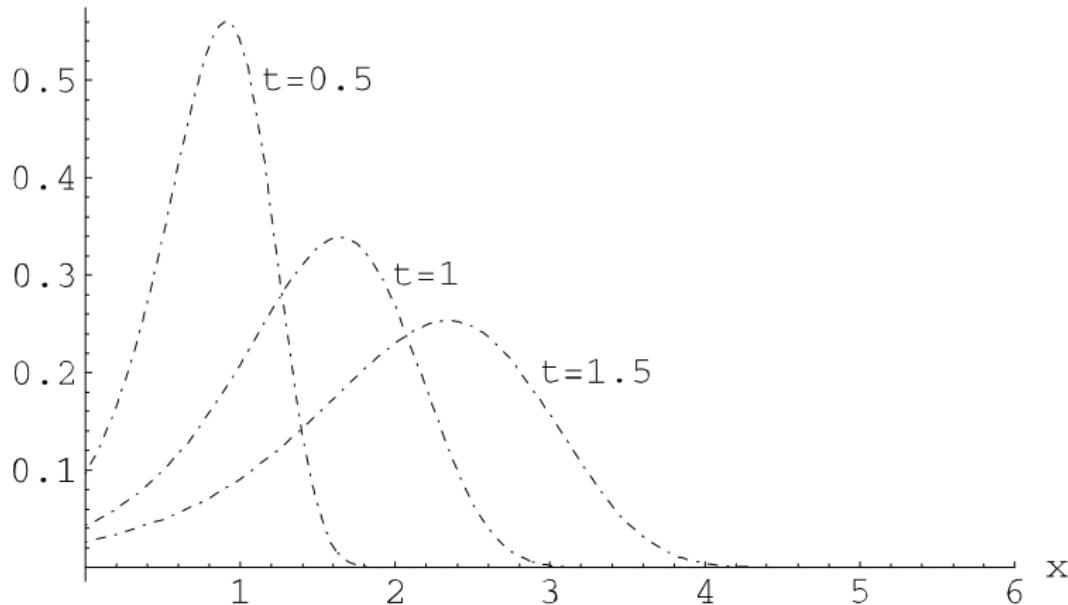
$u(x, t)$



Slika: Rešenje  $u(x, t)$ ,  $x \in (0, 3)$ ,  $t \in \{0.5, 1, 1.5\}$  za  $\alpha = 0.25$ .

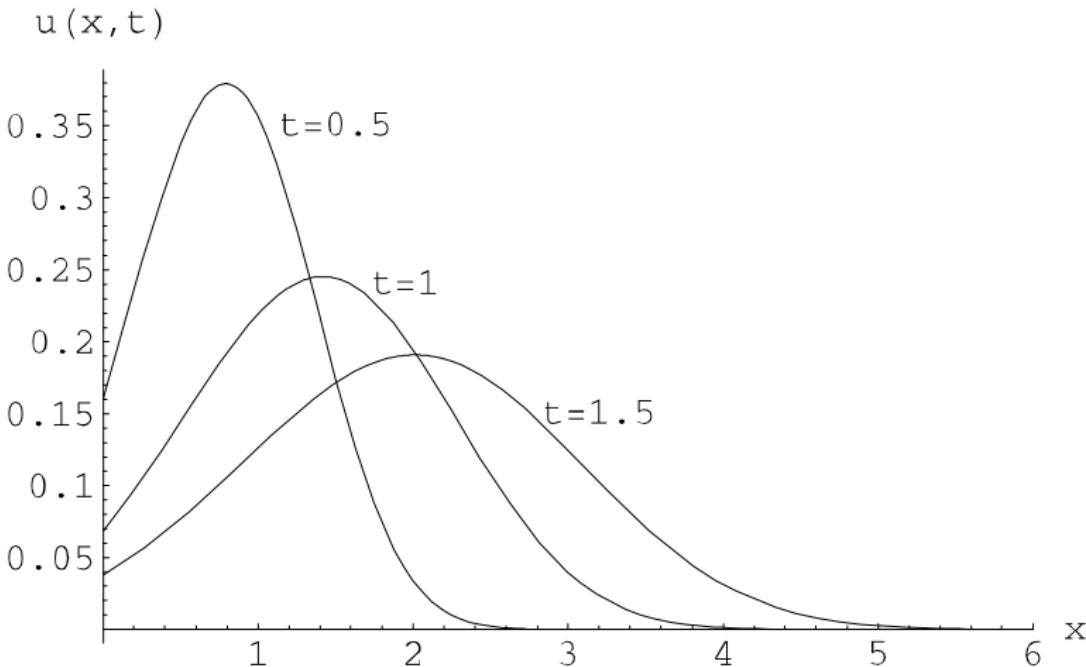
# Grafik rešenja $u$ za $u_0 = \delta$ i $v_0 = 0$ - II

$u(x, t)$



Slika: Rešenje  $u(x, t)$ ,  $x \in (0, 6)$ ,  $t \in \{0.5, 1, 1.5\}$  za  $\alpha = 0.5$ .

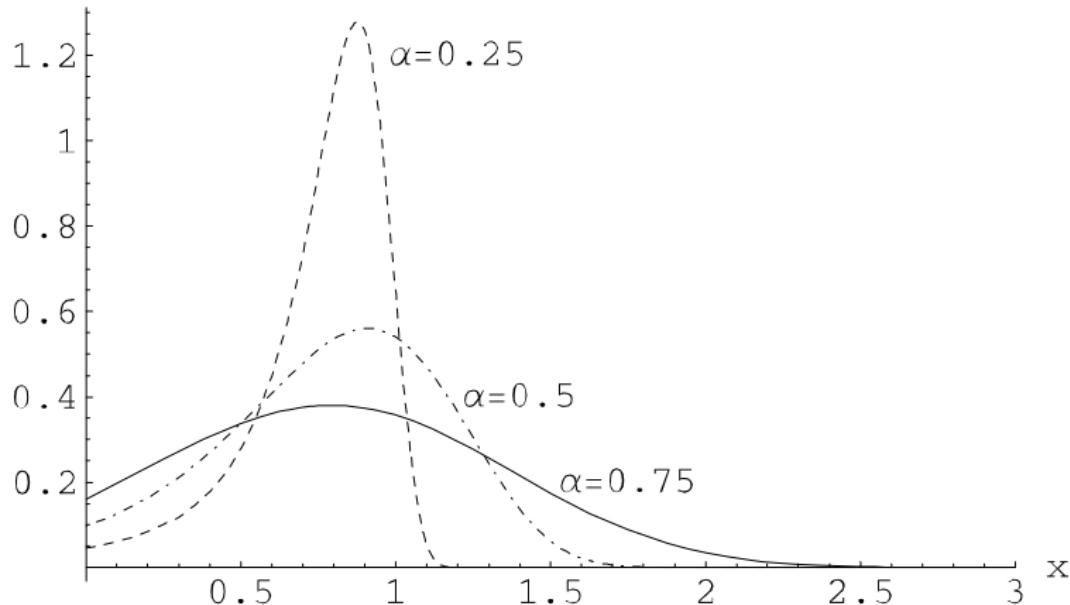
# Grafik rešenja $u$ za $u_0 = \delta$ i $v_0 = 0$ - III



Slika: Rešenje  $u(x, t)$ ,  $x \in (0, 6)$ ,  $t \in \{0.5, 1, 1.5\}$  za  $\alpha = 0.75$ .

# Grafik rešenja $u$ za $u_0 = \delta$ i $v_0 = 0$ - IV

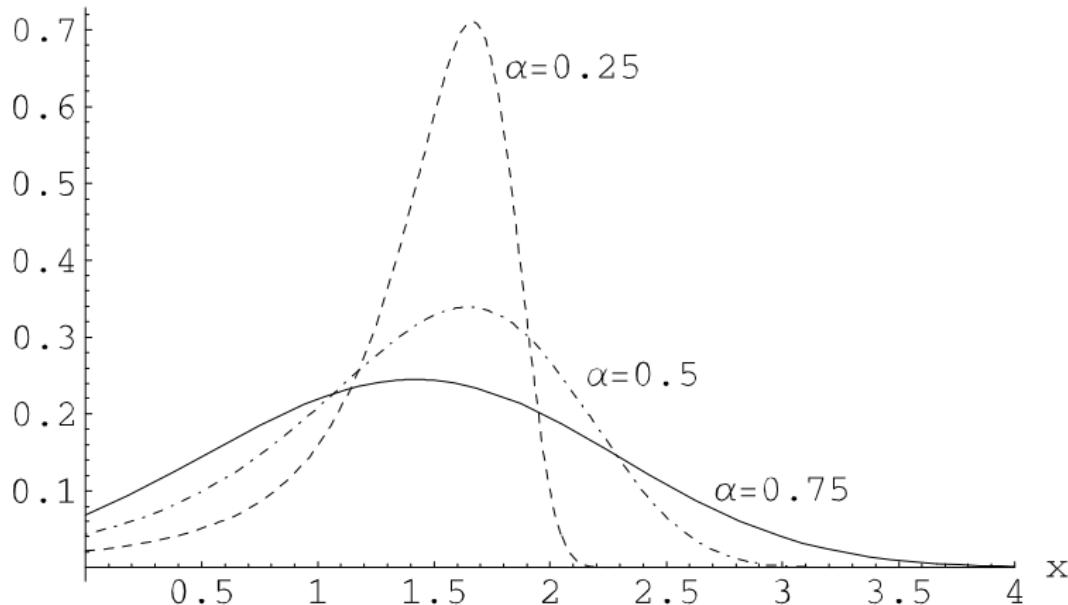
$u(x, 0.5)$



Slika: Rešenje  $u(x, t)$ ,  $x \in (0, 3)$ ,  $t = 0.5$  za  $\alpha \in \{0.25, 0.5, 0.75\}$ .

# Grafik rešenja $u$ za $u_0 = \delta$ i $v_0 = 0$ - V

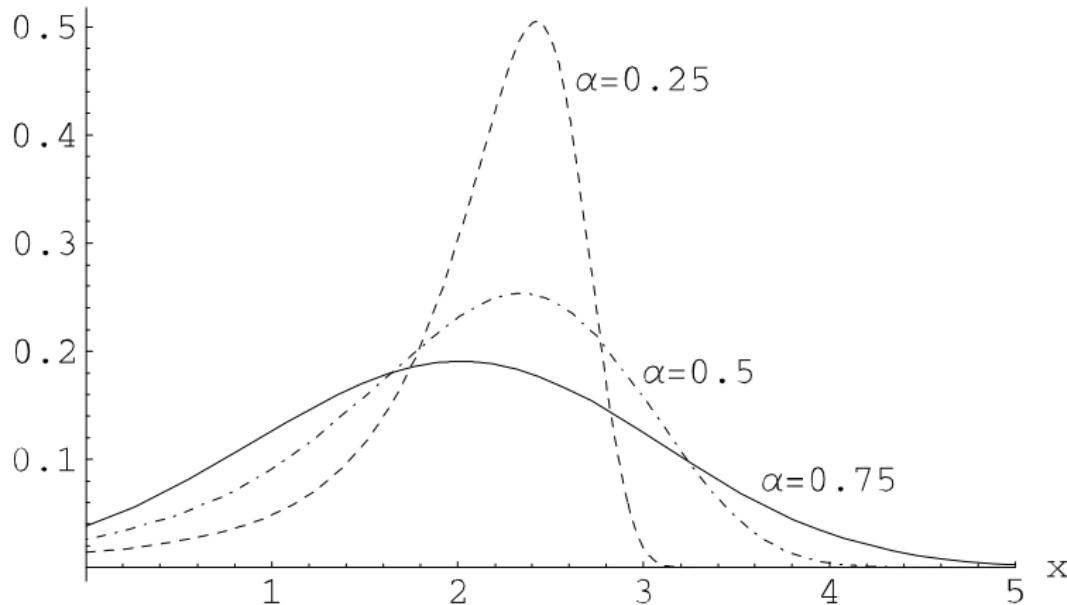
$u(x, 1)$



Slika: Rešenje  $u(x, t)$ ,  $x \in (0, 4)$ ,  $t = 1$  za  $\alpha \in \{0.25, 0.5, 0.75\}$ .

# Grafik rešenja $u$ za $u_0 = \delta$ i $v_0 = 0$ - VI

$u(x, 1.5)$



Slika: Rešenje  $u(x, t)$ ,  $x \in (0, 5)$ ,  $t = 1.5$  za  $\alpha \in \{0.25, 0.5, 0.75\}$ .

# Frakciona talasna jednačina linearnog tipa

Frakciona talasna jednačina linearnog tipa predstavljena je sistemom uz početne i granične uslove ( $x \in \mathbb{R}$ ,  $t > 0$ )

$$\frac{\partial}{\partial x} \sigma(x, t) = \frac{\partial^2}{\partial t^2} u(x, t), \quad \varepsilon(x, t) = \frac{\partial}{\partial x} u(x, t),$$

$$\sum_{k=0}^n a_k {}_0D_t^{\alpha_k} \sigma(x, t) = \sum_{k=0}^n b_k {}_0D_t^{\alpha_k} \varepsilon(x, t), \quad \frac{a_0}{b_0} \geq \frac{a_1}{b_1} \geq \dots \geq \frac{a_n}{b_n},$$

$$u(x, 0) = u_0(x), \quad \frac{\partial}{\partial t} u(x, 0) = v_0(x), \quad \sigma(x, 0) = 0, \quad \varepsilon(x, 0) = 0,$$

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} \sigma(x, t) = 0.$$

U distribucionoj postavci je oblika

$$\frac{\partial^2}{\partial t^2} u(x, t) = \mathcal{L}^{-1} \left[ \frac{\sum_{k=0}^n b_k s^{\alpha_k}}{\sum_{k=0}^n a_k s^{\alpha_k}} \right] *_t \frac{\partial^2}{\partial x^2} u(x, t) + u_0(x) \delta'(t) + v_0(x) \delta(t).$$

## Teorema

Neka su  $u_0, v_0 \in \mathcal{S}'(\mathbb{R})$ . Tada postoji jedinstveno rešenje  $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$  frakcione talasne jednačine Cenerovog tipa

$$u(x, t) = S(x, t) *_{x,t} (u_0(x)\delta'(t) + v_0(x)\delta(t)),$$

gde je

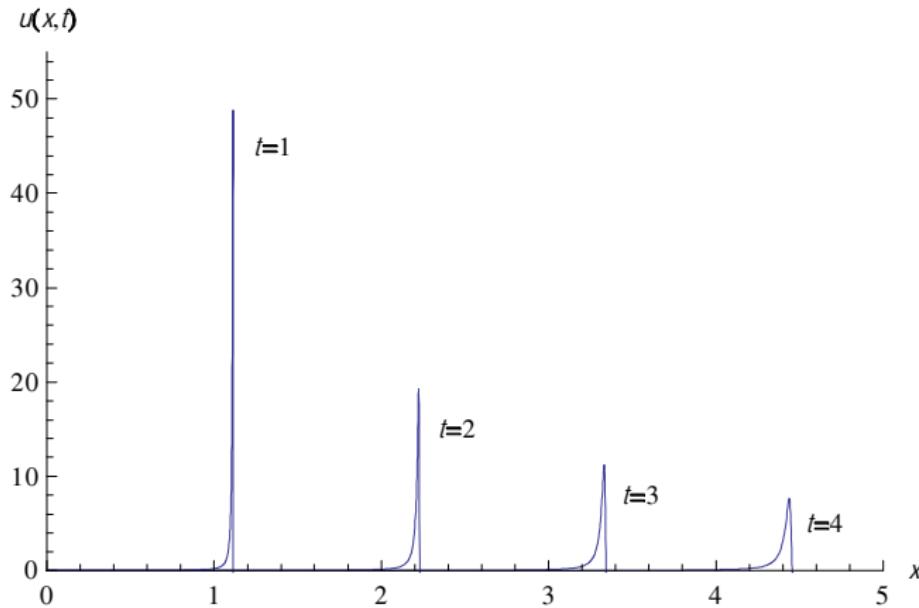
$$\begin{aligned} S(x, t) &= \frac{1}{2} \sqrt{\frac{a_0}{b_0}} + \frac{1}{4\pi i} \int_0^\infty \left( \sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} e^{i\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} e^{i\pi\alpha_k}}} e^{|x|q \sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} e^{i\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} e^{i\pi\alpha_k}}}} \right. \\ &\quad \left. - \sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} e^{-i\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} e^{-i\pi\alpha_k}}} e^{|x|q \sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k} e^{-i\pi\alpha_k}}{\sum_{k=0}^n b_k q^{\alpha_k} e^{-i\pi\alpha_k}}}} \right) \frac{e^{-qt}}{q} dq, \end{aligned}$$

fundamentalno rešenje,  $S \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}_+)$  sa nosačem u konusu  $|x| < ct$ ,

$$c = \sqrt{\frac{a_n}{b_n}}.$$

# Grafik rešenja $u$ za $u_0 = \delta$ i $v_0 = 0$ - I

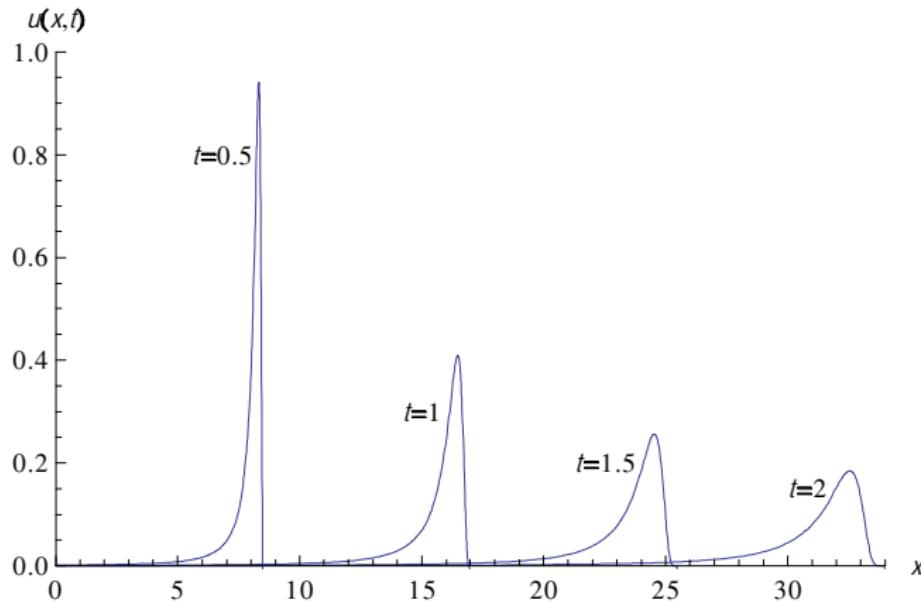
Neka su:  $\alpha_0 = 0.25$ ,  $\alpha_1 = 0.5$ ,  $\alpha_2 = 0.75$ ,  $a_0 = 1.25$ ,  $a_1 = 1.1$ ,  $a_2 = 1.2$ ,  $b_0 = 1.4$ ,  $b_1 = 1.3$ ,  $b_2 = 1.5$ .



Slika: Rešenje  $u(x, t)$ ,  $x \in (0, 5)$ ,  $t \in \{1, 2, 3, 4\}$ .

# Grafik rešenja $u$ za $u_0 = \delta$ i $v_0 = 0$ - II

Neka su:  $\alpha_0 = 0.25$ ,  $\alpha_1 = 0.5$ ,  $\alpha_2 = 0.75$ ,  $a_0 = 0.008$ ,  $a_1 = 0.006$ ,  
 $a_2 = 0.004$ ,  $b_0 = 1.6$ ,  $b_1 = 1.4$ ,  $b_2 = 1.2$ .



Slika: Rešenje  $u(x, t)$ ,  $x \in (0, 35)$ ,  $t \in \{0.5, 1, 1.5, 2\}$ .

# Frakciona talasna jednačina Eringenovog tipa

Frakciona talasna jednačina Eringenovog tipa predstavljena je sistemom ( $x \in \mathbb{R}$ ,  $t > 0$ )

$$\begin{aligned}\frac{\partial}{\partial x} \sigma(x, t) &= \frac{\partial^2}{\partial t^2} u(x, t), \quad \varepsilon(x, t) = \frac{\partial}{\partial x} u(x, t), \\ \sigma(x, t) - l_c^\alpha D_x^\alpha \sigma(x, t) &= E \varepsilon(x, t).\end{aligned}$$

Prepostavljajući rešenje frakcione talasne jednačine Eringenovog tipa u harmonijskom obliku

$$u(x, t) = u_0 e^{i(\omega t - kx)}, \quad \omega > 0, \quad k \in \mathbb{R}.$$

# Disperziona jednačina

- Dobija se disperziona jednačina

$$\omega(k) = \pm \frac{c_0 k}{\sqrt{1 - \cos \frac{\alpha\pi}{2} (I_c |k|)^\alpha}}, \quad c_0 = \sqrt{E/\rho}.$$

- Za  $k > 0$  disperziona jednačina

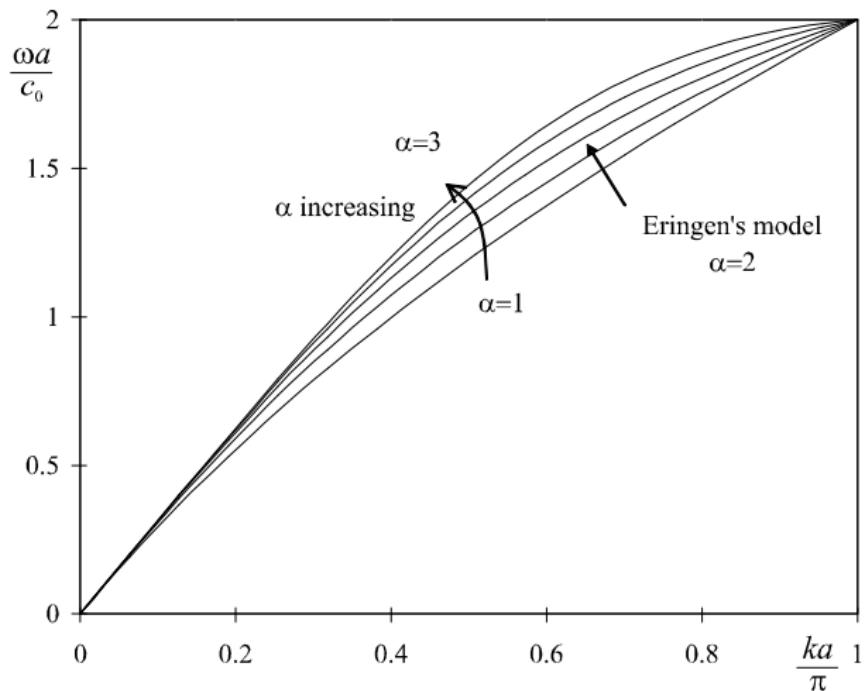
$$\frac{a}{c_0} \omega(k) = ka \frac{1}{\sqrt{1 - \left(\frac{I_c}{a}\right)^\alpha (ka)^\alpha \cos \frac{\alpha\pi}{2}}},$$

se poredi sa disperzionom jednačinom za Born-Karmanov model

$$\frac{a}{c_0} \omega_{bk}(k) = 2 \sin \frac{ka}{2}.$$

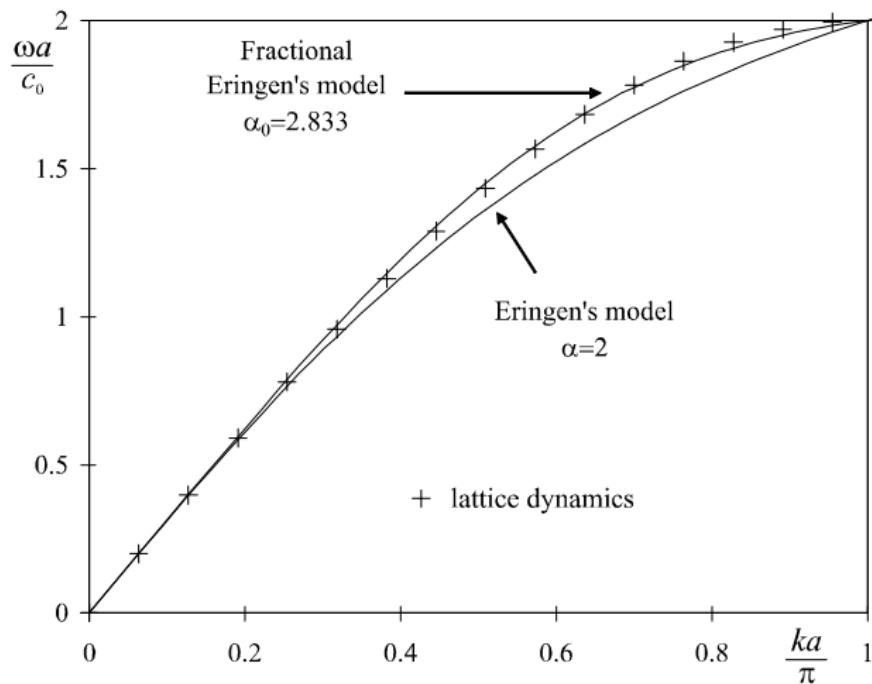
- Dobije se optimalni  $\alpha_0 = 2.833$  i  $I_c \cong 0.587a$  (za Eringenov model je  $I_c \cong 0.386a$ ).

# Grafik disperzije jednačine - I



Slika: Disperzije krive za frakcionu talasnu jedančinu Eringenovog tipa -  $\alpha \in \{1, 1.5, 2, 2.5, 3\}$ .

# Grafik disperzije jednačine - II



Slika: Poređenje disperzionih krivih.

# Prostorno-vremenska FTJ Cenerovog tipa

Prostorno-vremenska FTJ Cenerovog tipa predstavljena je sistemom uz početne i granične uslove ( $x \in \mathbb{R}$ ,  $t > 0$ )

$$\partial_x \sigma(x, t) = \partial_{tt} u(x, t), \quad \varepsilon(x, t) = \mathcal{E}_x^\beta u(x, t),$$

$$\sigma(x, t) + \tau_0 D_t^\alpha \sigma(x, t) = \varepsilon(x, t) + {}_0 D_t^\alpha \varepsilon(x, t), \quad \tau < 1,$$

$$u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = v_0(x), \quad \sigma(x, 0) = 0, \quad \varepsilon(x, 0) = 0,$$

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} \sigma(x, t) = 0.$$

U distribucionoj postavci je oblika

$$\partial_{tt} u(x, t) = L_t^\alpha \partial_x \mathcal{E}_x^\beta u(x, t) + u_0(x) \delta'(t) + v_0(x) \delta(t), \quad u \in \mathcal{K}'(\mathbb{R}^2),$$

$$L_t^\alpha = \mathcal{L}^{-1} \left[ \frac{1+s^\alpha}{1+\tau s^\alpha} \right] *_t = \left( \frac{1}{\tau} \delta(t) + \left( \frac{1}{\tau} - 1 \right) e_\alpha'(t) \right) *_t, \quad t > 0,$$

## Teorema

Neka su  $\alpha \in [0, 1)$ ,  $\beta \in [0, 1)$ ,  $\tau \in (0, 1)$  i neka je  $u_0, v_0 \in L^1(\mathbb{R})$ .

Tada postoji jedinstveno uopšteno rešenje  $u \in \mathcal{K}'(\mathbb{R}^2)$ ,

$\text{supp } u \subset \mathbb{R} \times [0, \infty)$ , prostorno-vremenske FTJ Cenerovog tipa sa početnim i graničnim uslovima, dato u obliku ( $x \in \mathbb{R}$ ,  $t > 0$ )

$$u(x, t) = \frac{1}{2\pi^2} (\delta'(t) u_0(x) + \delta(t) v_0(x)) *_{x,t} P(x, t),$$

$$P(x, t) = I(x, t) - \left( \frac{\partial}{\partial t} J_1(x, t) + \frac{\partial^2}{\partial t^2} J_2(x, t) \right) e^{s_0 t},$$

gde je

$$J_1 = i (J_1^+ - J_1^-), \quad J_2 = J_2^+ + J_2^-.$$

Funkcije  $I$ ,  $J_1^+$ ,  $J_1^-$ ,  $J_2^+$  i  $J_2^-$  su ograničene i neprekidne po  $x$  i neprekidne eksponencijalno ograničene po  $t$ .

## Teorema

Neka su svi uslovi prethodne teoreme zadovoljeni. Neka je  $u \in K'(\mathbb{R}^2)$ , sa nosačem u  $\mathbb{R} \times [0, \infty)$ , uopšteno rešenje prostorno-vremenske FTJ Cenerovog tipa sa početnim i graničnim uslovima, dato u obliku ( $x \in \mathbb{R}$ ,  $t > 0$ )

$$u(x, t) = (u_0(x)\delta(t) + v_0(x)H(t)) *_{x,t} K(x, t),$$

gde je  $K$  distribucioni limit u  $K'(\mathbb{R}^2)$ :

$$K(x, t) = \lim_{\varepsilon \rightarrow 0} K_\varepsilon(x, t),$$

$$K_\varepsilon(x, t) = \frac{1}{\pi} \int_0^\infty S(\rho, t) \cos(\rho x) e^{-\frac{(\varepsilon\rho)^2}{4}} d\rho,$$

gde je

$$S(\rho, t) = \frac{1}{2\pi i} \int_0^\infty \left( \frac{1}{q^2 + \frac{1+q^\alpha e^{i\alpha\pi}}{1+\tau q^\alpha e^{i\alpha\pi}} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} - \frac{1}{q^2 + \frac{1+q^\alpha e^{-i\alpha\pi}}{1+\tau q^\alpha e^{-i\alpha\pi}} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \right) q e^{-qt} dq$$

$$+ \frac{s e^{st}}{2s + \frac{\alpha(1-\tau)s^{\alpha-1}}{(1+\tau s^\alpha)^2} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \Bigg|_{s=s_Z(\rho)} + \frac{s e^{st}}{2s + \frac{\alpha(1-\tau)s^{\alpha-1}}{(1+\tau s^\alpha)^2} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \Bigg|_{s=\bar{s}_Z(\rho)}.$$

$$s_z \text{ su nule } \Psi_\alpha(s) = s^2 + \theta \frac{1+s^\alpha}{1+\tau s^\alpha}, \quad \theta = \rho^{1+\beta} \sin \frac{\beta\pi}{2}.$$

Za pogodno odabрано  $s_0 > 0$ ,  $K_\varepsilon(x, t) e^{-s_0 t}$  je ograničena i neprekidna po  $x \in \mathbb{R}$ ,  $t > 0$ , za svako  $\varepsilon \in (0, 1]$ .

# Specijalni slučajevi - I

- Polazeći od

$$u(x, t) = (u_0(x)\delta(t) + v_0(x)H(t)) *_{x,t} K_{\alpha,\beta}(x, t),$$

gde je

$$\tilde{K}_{\alpha,\beta}(x, s) = \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + \frac{1+s^\alpha}{1+\tau s^\alpha} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \cos(\rho x) d\rho.$$

- Ako  $\alpha \rightarrow 0$ , tada je

$$\tilde{K}_{0,\beta}(x, s) = \frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + \frac{2}{1+\tau} \rho^{1+\beta} \sin \frac{\beta\pi}{2}} \cos(\rho x) d\rho,$$

odnosno

$$K_{0,\beta}(x, t) = \frac{1}{\pi} \int_0^\infty \cos\left(t \sqrt{\frac{2}{1+\tau} \rho^{1+\beta} \sin \frac{\beta\pi}{2}}\right) \cos(\rho x) d\rho.$$

## Specijalni slučajevi - II

- U smislu distribucija je ( $c = \sqrt{2/(1+\tau)}$ )

$$K_{0,\beta}(x, t) = \frac{1}{2\pi} \int_0^\infty \left( \cos \left( \left( x + ct \sqrt{\frac{1}{\rho^{1-\beta}} \sin \frac{\beta\pi}{2}} \right) \rho \right) \right. \\ \left. + \cos \left( \left( x - ct \sqrt{\frac{1}{\rho^{1-\beta}} \sin \frac{\beta\pi}{2}} \right) \rho \right) \right) d\rho.$$

- Za  $\beta = 0$  dobija se

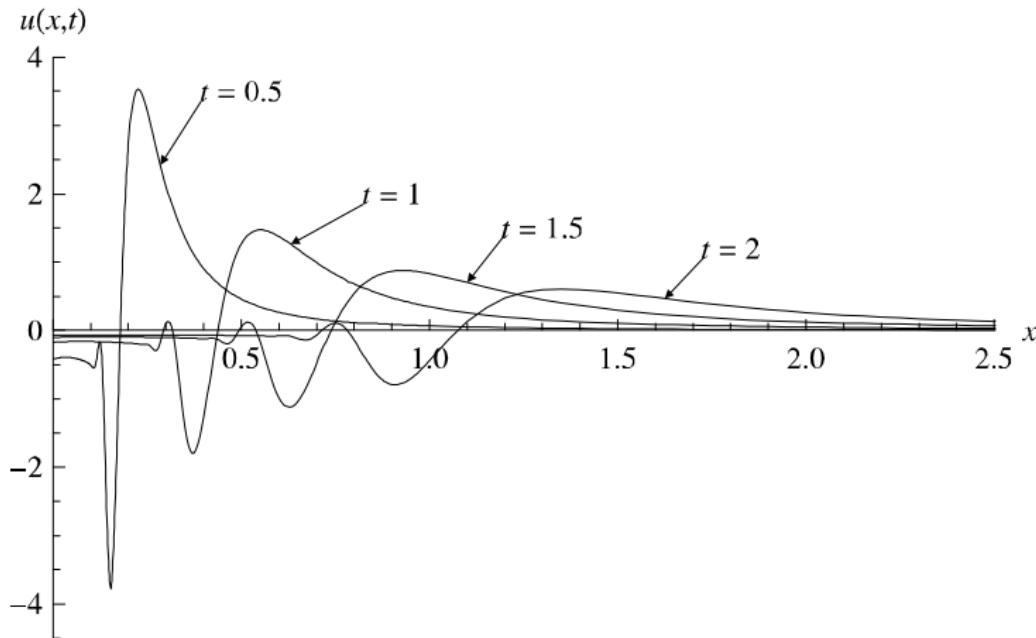
$$K_{0,0}(x, t) = \frac{1}{\pi} \int_0^\infty \cos(x\rho) d\rho = \delta(x).$$

- Za  $\beta = 1$  dobija se

$$K_{0,1}(x, t) = \frac{1}{2\pi} \int_0^\infty (\cos((x+ct)\rho) + \cos((x-ct)\rho)) d\rho, \\ = \frac{1}{2} (\delta(x+ct) + \delta(x-ct)).$$

# Grafik rešenja $u$ za $u_0 = \delta$ i $v_0 = 0$ - I

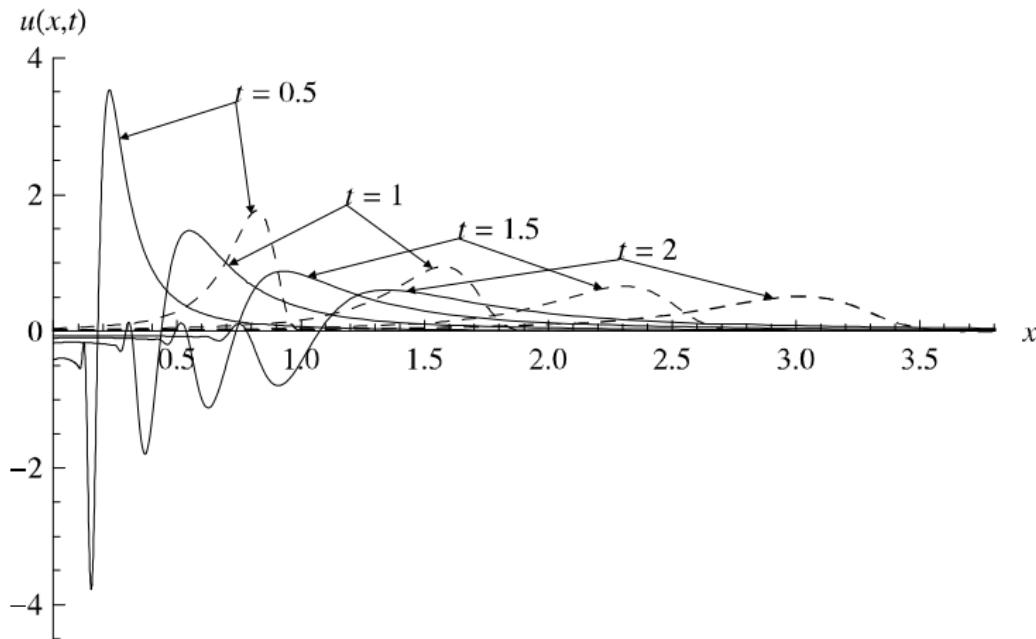
Neka su:  $\alpha = 0.25$ ,  $\beta = 0.45$ ,  $\tau = 0.1$ ,  $\varepsilon = 0.01$ .



Slika: Pomeranje  $u(x, t)$  za  $t \in \{0.5, 1, 1.5, 2\}$  kao funkcija  $x$ .

# Grafik rešenja $u$ za $u_0 = \delta$ i $v_0 = 0$ - II

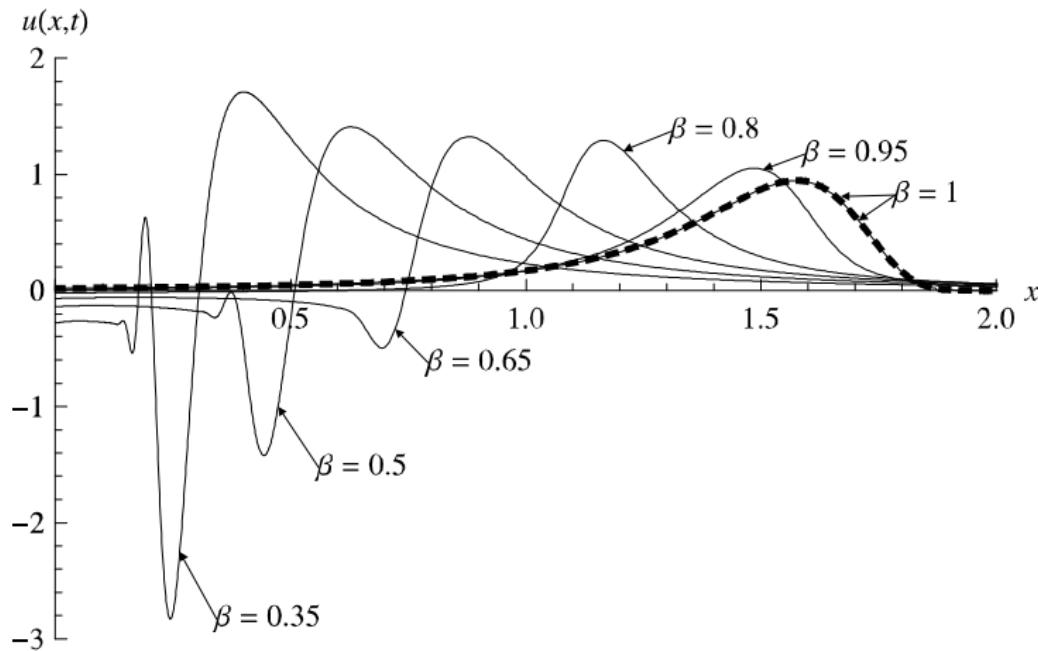
Neka su:  $\alpha = 0.25$ ,  $\tau = 0.1$ ,  $\varepsilon = 0.01$ .



Slika: Pomeranje  $u(x, t)$  za  $t \in \{0.5, 1, 1.5, 2\}$  kao funkcija  $x$ : puna linija -  $\beta = 0.45$ , isprekidana linija -  $\beta = 1$ .

# Grafik rešenja $u$ za $u_0 = \delta$ i $v_0 = 0$ - III

Neka su:  $\alpha = 0.25$ ,  $\tau = 0.1$ ,  $\varepsilon = 0.01$ .



Slika: Pomeranje  $u(x, t)$  za  $t = 1$  kao funkcija  $x$ .